The Gauss-Manin connection via the de Rham space

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ABSTRACT. We give a simple, geometric description of the Gauss-Manin connection in terms of the de Rham space X_{dR}.

CONTENTS

1. INTRODUCTION

Let k be a field of characteristic 0 and let $f\colon X \to S$ be a morphism of smooth k-schemes. It is a classical result that the relative algebraic de Rham cohomology $\mathrm{H}^i_{\mathrm{dR}}(X/S)$ carries a natural flat connection, the *Gauss–Manin* connection.

Analytically, we can see this as follows: suppose $k = \mathbb{C}$ and let $f^h : \mathfrak{X} \to \mathfrak{S}$ be the associated holomorphic map between smooth proper complex analytic varieties. Then, we have an isomorphism

$$\mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}) \cong \mathrm{R}^i f^h_* \mathbb{C}_{\mathfrak{X}} \otimes_{\mathbb{C}_{\mathfrak{S}}} \mathscr{O}_{\mathfrak{S}}$$

Assuming f^h is a proper submersion, $\mathrm{R}^i f^h_* \mathbb{C}_{\mathfrak{X}}$ is a locally constant sheaf by the Ehresmann fibration theorem. Then there is an analytic flat connection

$$(\mathrm{id} \otimes d) : \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}) \to \mathrm{H}^i_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{S}) \otimes \Omega^1_{\mathfrak{S}}$$

by the Riemann-Hilbert correspondence, which in turn gives a flat connection on $\mathrm{H}^i_{\mathrm{dR}}(X/S)$ by GAGA.

The algebraicity of the Gauss-Manin connection is more subtle. Katz and Oda provide one such proof of this by the observation that the connection appears as a differential d_1^{0i} on the E_1 page of a certain spectral sequence [KO68]. The aim of this note is to provide another such construction, this time coming from Simpson's de Rham space X_{dR}. In particular, we explain how the de Rham cohomology of X/S, considered as a $\mathscr{D}_S$-module under the Gauss-Manin connection, is isomorphic to the higher direct image of $\mathscr{O}_{X_{\mathrm{dR}}}$ along the induced morphism between de Rham spaces $f_{\mathrm{dR}} : X_{\mathrm{dR}} \to S_{\mathrm{dR}}$.

2. THE DE RHAM SPACE

2.1. **Preliminaries.** We introduce the de Rham space, originally defined by Simpson [Sim96]. Many of the technical arguments in this section are found in [Rib24].

Let k be a fixed field of characteristic 0 and Aff_k be the category of affine k-schemes. Let τ be a Grothendieck topology on Aff_k. We define a τ algebraic stack to be a sheaf $\mathrm{Aff}_k \to \mathrm{Grpd}$ in the τ topology. The category of τ algebraic stacks will be denoted AlgStk_τ.

Definition 2.1 (de Rham space)**.** Given any presheaf $X : \mathrm{Aff}_k \to \mathrm{Set}$, we define X_{dR} to be the presheaf

$$X_{\mathrm{dR}}(A) := X(A^{\mathrm{red}})$$

Note that there is a natural map $X \to X_{\mathrm{dR}}$ and that associated to a morphism $X \to S$ there is an induced morphism $X_{\mathrm{dR}} \to S_{\mathrm{dR}}$. In this case, we define the relative de Rham space $(X/S)_{\mathrm{dR}}$ to be the fibered product $X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S$.

Let X be a scheme which is formally smooth over k. If a k-algebra A has nilpotent nilradical (for instance, if A is Noetherian), then every A^{red}-point of X extends to an A-point,

$$\begin{tikzcd}
\operatorname{Spec} A^{\mathrm{red}} \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} k
\end{tikzcd}$$

by an infinitesimal lifting argument. Consequently, the natural map $X \to X_{\mathrm{dR}}$ is an epimorphism of presheaves. In general though, it suffices to have X/k smooth.

Proposition 2.2. *If X is smooth over k, then $X \to X_{\mathrm{dR}}$ is an epimorphism.*

Proof. Let $\mathfrak{J}$ be the poset of nilpotent ideals $I \subset A$. Then $A^{\mathrm{red}} \cong \varinjlim_{I \in \mathfrak{J}} A/I$, and

$$\mathrm{Hom}(\operatorname{Spec} A^{\mathrm{red}}, X) \cong \mathrm{Hom}(\varprojlim_{I \in \mathfrak{J}} \operatorname{Spec} A/I, X) \cong \varinjlim_{I \in \mathfrak{J}} \mathrm{Hom}(\operatorname{Spec} A/I, X)$$

where the second isomorphism holds because X is locally of finite presentation [Sta18] Tag 01ZC. Consider an element on the right; this is given by a representative $\operatorname{Spec} A/I \to X$ for some nilpotent ideal I. This lifts to $\operatorname{Spec} A \to X$ by the formal smoothness of X. $\square$

From the above proof, we also have the following characterization: if X is locally of finite presentation over k, then

$$X_{\mathrm{dR}}(A) \cong \mathrm{Hom}(\varprojlim_{I \in \mathfrak{J}} \operatorname{Spec} A/I, X) \cong \varinjlim_{I \in \mathfrak{J}} \mathrm{Hom}(\operatorname{Spec} A/I, X)$$

and therefore X_{dR} is a colimit. An important consequence of this is the following.

Corollary 2.3. *If X is locally of finite presentation over k, $X_{\mathrm{dR}} : \mathrm{Aff}_k \to \mathrm{Set}$ commutes with finite limits and arbitrary colimits.*

An important property of the de Rham space is that it satisfies descent.

Proposition 2.4 ([Rib24], Proposition 1.1.14)**.** *If X is a locally finite type scheme, the de Rham space X_{dR} satisfies fppf descent.*

We emphasize that k must have characteristic 0. Thus by our conventions, X_{dR} is an fppf algebraic stack.

2.2. **Algebraic de Rham cohomology.** A key feature of the de Rham space is that the cover $X \to X_{\mathrm{dR}}$ provides an identification of the algebraic de Rham cohomology of X with the $\mathscr{O}$-coherent cohomology of X_{dR}. We will prove a relative version of this theorem.

Let $f : X \to S$ be a morphism of k-schemes, and consider the induced morphisms

$$\begin{tikzcd}
(X/S)_{\mathrm{dR}} \arrow[r] \arrow[d, "\widetilde{f}"'] & X_{\mathrm{dR}} \arrow[d, "f_{\mathrm{dR}}"] \\
S \arrow[r] & S_{\mathrm{dR}}
\end{tikzcd}$$

Let $\mathscr{O}_{\mathrm{dR}/S}$ be the structure sheaf of $(X/S)_{\mathrm{dR}}$.

Theorem 2.5. *If $f : X \to S$ is a morphism of smooth k-schemes, then*

$$\mathrm{H}^i_{\mathrm{dR}}(X/S) \cong \mathrm{R}^i \widetilde{f}_* \mathscr{O}_{\mathrm{dR}/S}$$

where the left is the Zariski hypercohomology of $\Omega^\bullet_{X/S}$.

To establish 2.5, we need the following (both of which are proved in [Rib24]).

Lemma 2.6. *Let X be locally of finite presentation. Then, the formal completion of X along a closed subscheme Z may be identified as*

$$\widehat{X}_Z \cong X \times_{X_{\mathrm{dR}}} Z_{\mathrm{dR}}$$

Lemma 2.7. *If X and S are locally of finite presentation,*

$$X \times_{(X/S)_{\mathrm{dR}}} X \cong (\widehat{X \times_S X})_\Delta$$

where the right-hand side is the formal completion of the diagonal $X \to X \times_S X$.

Some remarks are in order. A priori, the definition of formal completion of X along a subscheme Z requires Z to be closed. However in 2.6, note that the right-hand side does not require $Z \to X$ to be a closed immersion. Thus in 2.7, we do not require $X \to S$ to be separated and instead define formal completion as in 2.6.

Proof of 2.7. Let $Y = X \times_S X$, so that $\widehat{Y}_\Delta \cong Y \times_{Y_{\mathrm{dR}}} X_{\mathrm{dR}}$. The result follows from

$$\begin{tikzcd}
X \times_{(X/S)_{\mathrm{dR}}} X \arrow[r] \arrow[d] & Y = X \times_S X \arrow[d] \\
X_{\mathrm{dR}} \arrow[r, "\Delta"'] & Y_{\mathrm{dR}} \cong X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} X_{\mathrm{dR}}
\end{tikzcd}$$

being cartesian. ☐

Proof of 2.5. For simplicity, take $S = \operatorname{Spec} k$. Since $\widehat{X \times X} \rightrightarrows X \to X_{\mathrm{dR}}$ is a coequalizer of presheaves, $X \to X_{\mathrm{dR}}$ is an effective epimorphism. Therefore the simplicial presheaf $\widehat{X}^\bullet = \check{\mathrm{C}}\mathrm{ech}(X \to X_{\mathrm{dR}})$ is a hypercover of X_{dR}. This gives

$$\mathrm{H}^i(X_{\mathrm{dR}}, \mathscr{O}_{X_{\mathrm{dR}}}) \cong \mathrm{H}^i(\widehat{X}^\bullet, \mathscr{O}_{\widehat{X}^\bullet})$$

By 2.7, $\widehat{X}^n$ is the ind-scheme given by formal completion of the n-fold product $X \times \cdots \times X$ along the diagonal. This simplicial scheme is used by Grothendieck to calculate the cohomology of the *infinitesimal site* in [Gro68]. This hinges on the observation that $\widehat{X}^\bullet$ gives a hypercover of the terminal object in the infinitesimal topos. Precisely, there is a spectral sequence

$$\mathrm{E}_2^{ij} = \mathrm{H}^i(n \mapsto \mathrm{H}^j(\widehat{X}^{n+1}, \mathscr{O}_{\widehat{X}^{n+1}})) \implies \mathrm{H}^{i+j}(X_{\mathrm{inf}}, \mathscr{O}_{X_{\mathrm{inf}}})$$

which degenerates and yields a canonical isomorphism

$$\mathrm{H}^i(X_{\mathrm{inf}}, \mathscr{O}_{X_{\mathrm{inf}}}) \cong \mathrm{H}^i(n \mapsto \mathscr{O}(\widehat{X}^{n+1})) = \mathrm{H}^i(\widehat{X}^\bullet, \mathscr{O}_{\widehat{X}^\bullet})$$

because $\mathrm{H}^j(\widehat{X}^{n+1}, \mathscr{O}_{\widehat{X}^{n+1}}) = 0$ for $j > 0$. It's a deeper result that for smooth schemes in characteristic 0, there is a canonical isomorphism

$$\mathrm{H}^i(X_{\mathrm{inf}}, \mathscr{O}_{X_{\mathrm{inf}}}) \cong \mathrm{H}^i_{\mathrm{dR}}(X/k)$$

(Theorem 4.1 in [Gro68]). The exact same proof works over a general base S. ☐

2.3. **Quasi-coherent sheaves on de Rham spaces.** A key insight of Simpson was that for a smooth k-scheme X, the category of quasi-coherent sheaves on X_{dR} identifies with the category of algebraic left $\mathscr{D}_X$-modules on X. Since

$$\widehat{X \times X} \rightrightarrows X \to X_{\mathrm{dR}}$$

is a coequalizer of presheaves, we may consider X_{dR} as a quotient of X by the relation of *infinitesimal closeness.* The data of a quasi-coherent sheaf on X_{dR} is thus equivalent to a quasi-coherent sheaf on X along with a compatible system of isomorphisms over points which lie in some formal thickening of the diagonal. This is the definition of a *crystal of quasi-coherent sheaves* on X; such objects are equivalent to $\mathscr{D}_X$-modules.

Theorem 2.8. *If X is a smooth k-scheme, there is an equivalence*

$$\Phi : \mathrm{QCoh}(X_{\mathrm{dR}}) \to \mathscr{D}_X\text{-mod}$$

between the category of quasi-coherent sheaves on X_{dR} and the category of quasi-coherent left $\mathscr{D}_X$-modules.

For a proof of this statement, see [Sta18] Tag 07J6. This equivalence extends to a *cohomological* equivalence. Given a left $\mathscr{D}_X$-module $(\mathscr{F}, \nabla)$ on X, define the de Rham complex $\mathrm{dR}_X(\mathscr{F})$ to be the complex

$$\mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_X} \Omega^1_X \to \mathscr{F} \otimes_{\mathscr{O}_X} \Omega^2_X \to \cdots$$

obtained by iterating ∇. Then the hypercohomology of this complex is isomorphic to the cohomology of the associated quasi-coherent sheaf on X_{dR}.

Proposition 2.9. *If X is a smooth k-scheme and $(\mathscr{F}, \nabla)$ is a left $\mathscr{D}_X$-module,*

$$\mathbb{H}^i(X, \mathrm{dR}_X(\mathscr{F})) \cong \mathrm{H}^i(X_{\mathrm{dR}}, \Phi(\mathscr{F}, \nabla))$$

Proof. Follow the proof of 2.5 with $\Phi(\mathscr{F}, \nabla)$ in place of $\mathscr{O}_{X_{\mathrm{dR}}}$. □

Let S be a smooth k-scheme. Then 2.8 gives an equivalence of functors $F \cong \pi^*$ where F is the forgetful functor $\mathscr{D}_S\text{-mod} \to \mathrm{QCoh}(S)$ and π is the natural morphism $S \to S_{\mathrm{dR}}$. Therefore, any quasi-coherent sheaf $\mathscr{F}$ on S can be endowed with a flat connection precisely when it is in the essential image of π^*. Then the fact that the relative de Rham cohomology sheaf $\mathrm{H}^i_{\mathrm{dR}}(X/S)$ has a flat connection must correspond to the fact that it is pulled back from some quasi-coherent sheaf on S_{dR}.

Consider the diagram,

$$\begin{tikzcd}
(X/S)_{\mathrm{dR}} \arrow[r, "\widetilde{\pi}"] \arrow[d, "\widetilde{f}"'] & X_{\mathrm{dR}} \arrow[d, "f_{\mathrm{dR}}"] \\
S \arrow[r, "\pi"'] & S_{\mathrm{dR}}
\end{tikzcd} \tag{$*$}$$

and recall we have a natural map

$$\pi^* \mathrm{R}^i f_{\mathrm{dR}\,*} \mathscr{O}_{X_{\mathrm{dR}}} \to \mathrm{R}^i \widetilde{f}_* \widetilde{\pi}^* \mathscr{O}_{X_{\mathrm{dR}}} \cong \mathrm{H}^i_{\mathrm{dR}}(X/S)$$

called the base-change morphism.

Proposition 2.10. *Let $f : X \to S$ be a morphism of smooth k-schemes. If $(*)$ satisfies base change, there exists a canonical flat connection ∇ on $\mathrm{H}^i_{\mathrm{dR}}(X/S)$.*

The next section is devoted to the proof of this base-change formula.

Because of the non-algebraic nature of S_{dR}, establishing the above base change is slightly delicate; since S_{dR} is not a scheme, we cannot apply traditional proper or smooth base change for quasi-coherent sheaves on schemes. To this end, we proceed by setting up some abstract machinery: we appeal to the 6-functor formalism for quasi-coherent sheaves on X_{dR} as found in [Rib24] and [Rod25]. In this section, assume that all functors are appropriately derived.

3.1. Abstract 6-functor formalisms.

We recall the definition of a 6-functor formalism. Rather than strive for an exhaustive treatment, we will introduce only the minimal amount of theory to state the desired result. For details see [Sch22].

Definition 3.1. A *geometric setup* is a pair $(\mathcal{C}, E)$ where $\mathcal{C}$ is an ∞-category with finite limits and E is a collection of morphisms in $\mathcal{C}$ containing all isomorphisms and stable under pullback, composition, and taking diagonals.

Remark 3.2. E is the class of !-able morphisms, or morphisms which admit an $f_!$.

Definition 3.3. Given a geometric setup $(\mathcal{C}, E)$, the category of correspondences $\mathrm{Corr}(\mathcal{C}, E)$ is defined to be the category whose objects are the objects of $\mathcal{C}$ and a morphism from X to Y is a diagram

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

in $\mathcal{C}$ where the arrow on the right $Z \to Y$ lies in E. A composition of morphisms is

$$\begin{tikzcd} & & Z \times_Y S \arrow[dl] \arrow[dr] & & \\ & Z \arrow[dl] \arrow[dr] & & S \arrow[dl] \arrow[dr] & \\ X & & Y & & T \end{tikzcd}$$

where the left arrow $Z \times_Y S \to T$ lies in E by assumption. There is a monoidal structure on $\mathrm{Corr}(\mathcal{C}, E)$ given by the product in $\mathcal{C}$.

Definition 3.4. Given a geometric setup $(\mathcal{C}, E)$, a 3-functor formalism is a lax symmetric monoidal functor $\mathcal{D} : \mathrm{Corr}(\mathcal{C}, E) \to \mathrm{Cat}_\infty$.

Given an arbitrary morphism $f : X \to Y$ in $\mathcal{C}$, the image of the correspondence

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, equal] & \\ Y & & X \end{tikzcd}$$

is defined to be $f^* : \mathcal{D}(Y) \to \mathcal{D}(X)$. If $f : X \to Y$ lies in E, the image of

$$\begin{tikzcd} & X \arrow[dl, equal] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

is defined to be $f_! : \mathcal{D}(X) \to \mathcal{D}(Y)$.

Definition 3.5. A 6-functor formalism is a 3-functor formalism

$$\mathcal{D} : \mathrm{Corr}(\mathcal{C}, E) \to \mathrm{Cat}_\infty$$

such that the following holds:

(a) the functors f^* and $f_!$ both admit right adjoints (defined to be f_* and $f^!$),

(b) for each object of $\mathcal{C}$, the symmetric monoidal ∞-category $\mathcal{D}(X)$ is closed.

A formal consequence of this is the following: given any Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

in $\mathcal{C}$ where f is !-able, there is a specified isomorphism of functors

$$g^* \circ f_! \to f'_! \circ (g')^*$$

[Man22]. This is a salient feature of 6-functor formalisms and the motivation for our consideration of them.

Definition 3.6. Let $f : X \to Y$ be !-able and $\omega_f := f^!(1_Y)$ be the dualizing object. The morphism f is said to be *cohomologically smooth* provided that

(a) The natural transformation $\omega_f \otimes f^*(-) \to f^!(-)$ is an equivalence,

(b) ω_f is $\otimes$-invertible in $\mathcal{D}(X)$,

(c) For any morphism $g : Y' \to Y$ with base change $f' : X' \to Y'$ of f, the above properties also hold for f'. Moreover, if $g' : X' \to X$ denotes the base change of g, we also require the natural map

$$(g')^*\omega_f \to \omega_{f'}$$

to be an isomorphism.

For more details see [Sch22] §5. Cohomological smoothness is important because it yields an abstract notion of smooth base change ([Zav23], lemma 2.3.9).

Theorem 3.7 (Cohomologically smooth base change)**.** *Let*

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"'] & Y
\end{tikzcd}$$

be a Cartesian square in $\mathcal{C}$*. Then the natural morphism*

$$g^* \circ f_* \to (f')_* \circ (g')^*$$

is an isomorphism if g *is cohomologically smooth.*

As a final point, we mention briefly a particular type of cohomologically smooth morphism called cohomologically étale morphisms. Intuitively, cohomologically étale morphisms are cohomologically smooth with trivial dualizing object, i.e. satisfying $f^* \cong f^!$. We do not include a precise definition since cohomological smoothness is a sufficient condition for our needs (see [Zav23] for a precise definition).

3.2. **Quasi-coherent sheaves on algebraic stacks.** We outline a construction for a 6-functor formalism for the derived category of quasi-coherent sheaves on algebraic stacks. This relies on extension theorems for 6-functor formalisms which we will treat as a black box. This is done in full detail in [Rod25] and [Rib24].

As before, let Aff_k be the category of affine k-schemes.

Theorem 3.8. *The functor* $\mathrm{D}_{\mathrm{qc}} : \mathrm{Aff}_k^{\mathrm{op}} \to \mathrm{Cat}_\infty$ *which maps* X *to its derived* ∞*-category of quasi-coherent sheaves* $\mathrm{D}_{\mathrm{qc}}(X)$ *extends to a 6-functor formalism on the geometric setup* $(\mathrm{Aff}_k, \mathrm{all})$ *consisting of all morphisms.*

This is proved in [Sch22] §8.3. To extend this to a 6-functor formalism on algebraic stacks, we appeal to results of [HM24]. This uses the D_{qc}-topology, though we will not elaborate on this since we can restrict to the coarser fppf topology.

Theorem 3.9. *The 6-functor formalism on the geometric setup* $(\mathrm{Aff}_k, \mathrm{all})$ *extends uniquely to a 6-functor formalism* $(\mathrm{AlgStk}_{\mathrm{D}_{\mathrm{qc}}}, E)$ *such that*

(a) Let $f : Y \to X$ *be a map whose pullback to every object in* Aff_k *lies in* E*, then* f *lies in* E*.*

(b) Let $f : Y \to X$ *be a map which is* $!$*-locally on the source or target in* E*; then* f *lies in* E*.*

(c) Every map $f : Y \to X$ *in* E *with* X *a scheme is* $!$*-locally on the source in* E*.*

We emphasize that we do not have explicit knowledge of the class E. To check that a particular morphism of algebraic stacks is !-able, we need to verify one of the above conditions. For a detailed account, see [Sch22] *Appendix to Lecture IV: Passage to stacks* and [Rod25], Corollary 6.19.

Remark 3.10. The above result uses the D_{qc}-topology on algebraic stacks. However this does not pose a substantial problem; the D_{qc}-topology is finer than the fppf topology, and [Rib24] shows that there is a unique 6-functor formalism on fppf algebraic stacks whose !-able morphisms are precisely the morphisms whose sheafifications lie in the class E above (proposition 1.3.23). That is, there is a compatible 6-functor formalism on the geometric setup $(\mathrm{AlgStk}_{\mathrm{fppf}}, E')$. In the sequel, we restrict to this formalism.

We need the following technical lemmas to proceed. Let X and Y be schemes.

Lemma 3.11 ([Sch22], Proposition 8.14)**.** *If $f : X \to Y$ is smooth and proper, then f is cohomologically smooth.*

Lemma 3.12 ([Rib24], Proposition 1.3.25)**.** *Let $Z \subset X$ be a closed subscheme, where the ideal sheaf is locally finitely generated. Then the morphism*

$$\widehat{X}_Z \to X$$

is cohomologically étale.

With these, we can prove the following central lemma.

Proposition 3.13. *If X is smooth over k, the natural morphism $X \to X_{\mathrm{dR}}$ is !-able and cohomologically smooth.*

We remark that the proof of this result is essentially ([Sch22], proposition 8.23), but Scholze works in the IndCoh-formalism.

Proof. Cohomological smoothness can be verified locally on the source and target. Since this is local on the source, we can restrict to an affine U. Moreover, by smoothness, we can assume U is of finite presentation over k. By taking a compactification, we can restrict to a proper U. Let $Y \to U_{\mathrm{dR}}$ be a morphism with Y locally of finite presentation over k. Then the following diagram is Cartesian,

$$\begin{tikzcd}
(\widehat{U \times_k Y})_Z \arrow[r] \arrow[d] & Y \arrow[d] \\
U \arrow[r] & U_{\mathrm{dR}}
\end{tikzcd}$$

where $Z \subset U \times_k Y$ is the graph of $Y_{\mathrm{red}} \to U$, and Z is closed because U is separated and $Y^{\mathrm{red}} \to Y$ is a closed immersion. Then $(\widehat{U \times_k Y})_Z \to Y$ is the composite of $(\widehat{U \times_k Y})_Z \to U \times_k Y$ and $U \times_k Y \to Y$. Since the ideal sheaf of Z is finitely generated, the first morphism is cohomologically smooth by 3.12. The second is a base change of $U \to \operatorname{Spec} k$ and is thus cohomologically smooth by 3.11. □

Proposition 3.14. *Let $f : X \to Y$ be a morphism between smooth, finite type k-schemes. Then $f_{\mathrm{dR}} : X_{\mathrm{dR}} \to Y_{\mathrm{dR}}$ is !-able in the 6-functor formalism for quasi-coherent sheaves.*

Proof. Since the class of !-able morphisms is stable under composition, it suffices to verify this after precomposing with the cohomologically smooth map $X \to X_{\mathrm{dR}}$. Then $X \to Y_{\mathrm{dR}}$ is the composite of f and $Y \to Y_{\mathrm{dR}}$, which are both !-able. □

3.3. **The Gauss-Manin connection.** Let $f : X \to S$ be a morphism of smooth, finite type k-schemes and consider the diagram

$$\begin{tikzcd}
(X/S)_{\mathrm{dR}} \arrow[r, "\widetilde{\pi}"] \arrow[d, "\widetilde{f}"'] & X_{\mathrm{dR}} \arrow[d, "f_{\mathrm{dR}}"] \\
S \arrow[r, "\pi"'] & S_{\mathrm{dR}}
\end{tikzcd} \tag{$*$}$$

Then, 3.7 and 3.13 imply the following:

Proposition 3.15. *The natural transformation*

$$\pi^* \mathrm{R}^i f_{\mathrm{dR}\,*} \to \mathrm{R}^i \widetilde{f}_* \widetilde{\pi}^*$$

is an equivalence.

We emphasize that the smoothness of f is unnecessary to establish base change for the relative de Rham square; since π is cohomologically smooth we need not assume any conditions on f_{dR} (other than that it's !-able).

Corollary 3.16. *There is a canonical flat connection on the relative de Rham cohomology* $\mathrm{H}^i_{\mathrm{dR}}(X/S)$.

This $\mathscr{D}_S$-module is naturally isomorphic to the i-th higher direct image of the structure sheaf of X_{dR} along f_{dR},

$$(\mathrm{H}^i_{\mathrm{dR}}(X/S), \nabla) \cong \mathrm{R}^i f_{\mathrm{dR}\,*} \mathscr{O}_{X_{\mathrm{dR}}}$$

Remark 3.17. We can also state the following slightly more general version: let $(\mathscr{F}, \nabla)$ be a left $\mathscr{D}_X$-module, and let $\mathrm{dR}_{X/S}(\mathscr{F}) = \mathscr{F} \otimes_{\mathscr{O}_X} \Omega^\bullet_{X/S}$ be the relative de Rham complex. By the relative version of 2.9, we have an isomorphism

$$\mathbb{R}^i f_* (\mathrm{dR}_{X/S}(\mathscr{F})) \cong \pi^* \mathrm{R}^i f_{\mathrm{dR}\,*} \mathscr{F}'$$

where $\mathscr{F}' = \Phi(\mathscr{F}, \nabla)$ is the quasi-coherent sheaf on X_{dR} corresponding to $(\mathscr{F}, \nabla)$. Consequently the right-hand side has a natural flat connection, which generalizes the Gauss-Manin connection to "de Rham cohomology" with coefficients in $(\mathscr{F}, \nabla)$.

Remark 3.18. This also provides a compact proof of the spectral sequence

$$\mathbb{H}^i(S, \mathrm{dR}_S(\mathrm{H}^j_{\mathrm{dR}}(X/S))) \implies \mathrm{H}^{i+j}_{\mathrm{dR}}(X/k)$$

described by [KO68]. This is the Leray spectral sequence for $f_{\mathrm{dR}} : X_{\mathrm{dR}} \to S_{\mathrm{dR}}$

$$\mathrm{H}^i(S_{\mathrm{dR}}, \mathrm{R}^j f_{\mathrm{dR}\,*} \mathscr{O}_{X_{\mathrm{dR}}}) \implies \mathrm{H}^{i+j}(X_{\mathrm{dR}}, \mathscr{O}_{X_{\mathrm{dR}}})$$

by 2.9.

4. Appendix: generalities on cohomology theories

The Gauss-Manin connection on $\mathrm{H}^i_{\mathrm{dR}}(X/S)$ is a particular case of a general construction. Let k be a field and τ a Grothendieck topology on Aff_k. We make the following nonstandard definition.

Definition 4.1. An R-cohomology theory is a cohomology theory for some class $\mathcal{S}$ of k-schemes with an associated 6-functor formalism D_R satisfying the following: For any $X \in \mathcal{S}$, there is a functorial assignment $X \rightsquigarrow X_R$, where the latter is a τ algebraic stack along with a morphism $X \to X_R$. This data is required to induce

(a) an isomorphism of cohomology,

$$\mathrm{H}^i_R(X/k) \cong \mathrm{R}^i\Gamma(X_R, \mathscr{O})$$

and likewise for relative cohomology with $(X/S)_R := X_R \times_{S_R} S$,

(b) and an equivalence of 6-functor formalisms,

$$\mathrm{D}_R(X) \cong \mathrm{D}_{\mathrm{qc}}(X_R)$$

In this context, we often call D_R the coefficients for the cohomology theory H_R.

The terminology comes from the fact that in many cases there is a *ring stack* $\mathcal{R}$ such that $X_R \cong X \circ \mathcal{R}$. In the case of de Rham cohomology, this is $\mathbb{A}^1_{\mathrm{dR}}$. Indeed, this is the functor on k-algebras defined by $A \mapsto A^{\mathrm{red}}$ and hence $X_{\mathrm{dR}}(A) = X(\mathbb{A}^1_{\mathrm{dR}}(A))$.

Example 4.2. De Rham, Dolbeault, crystalline, and prismatic cohomology are all examples of R-cohomology theories for some suitable class of k-schemes, with associated coefficients $\mathscr{D}$-modules, Higgs sheaves, crystals, and F-gauges, respectively. Moreover, in each of these cases, $X_R = X \circ \mathbb{A}^1_{\mathrm{R}}$.

Let $f : X \to S$ be a morphism in $\mathcal{S}$ and consider the induced $f_R : X_R \to S_R$.

Proposition 4.3. *Suppose $X \to X_R$ is cohomologically smooth for D_{qc}. Then the relative cohomology $\mathrm{H}^i_R(X/S)$ has a canonical R-coefficient structure via $\mathrm{R}^i(f_R)_*\mathscr{O}_{X_R}$.*

To summarize, we always have a natural additional structure on relative cohomology induced from $X \rightsquigarrow X_R$.

4.1. The Kodaira-Spencer map.

Let k be a field of characteristic 0 and X a smooth k-scheme. Recall that the *algebraic Dolbeault cohomology* is defined to be

$$\mathrm{H}^i_{\mathrm{Dol}}(X/k) := \bigoplus_{p+q=i} \mathrm{R}^p f_* \Omega^q_{X/k}$$

Dolbeault cohomology is an R-cohomology theory in the sense of 4.1 [Sim99]. That is, there exists an fppf algebraic stack X_{Dol} and a morphism $X \to X_{\mathrm{Dol}}$ whose

$\mathscr{O}$-coherent cohomology recovers $\mathrm{H}_{\mathrm{Dol}}$. The category of quasi-coherent sheaves on X_{Dol} is equivalent to the category of quasi-coherent *Higgs* sheaves on X,

$$\mathrm{QCoh}(X_{\mathrm{Dol}}) \cong \mathrm{Higgs}(X)$$

which we view as the coefficient objects for Dolbeault cohomology.

The stack X_{Dol} has a simplicial presentation as $\widehat{T^{\bullet}X}$, where T^nX is the n-fold product $TX \times_X \cdots \times_X TX$ and the formal completion is along the zero section $X \to T^nX$. We can copy the proof of 3.13 to conclude the following,

Proposition 4.4. *If X is smooth over k, $X \to X_{\mathrm{Dol}}$ is cohomologically smooth.*

Suppose we have a morphism $f : X \to S$ of smooth, finite type k-schemes. Then 4.3 implies that there is a natural Higgs field ϑ on $\mathrm{H}^i_{\mathrm{Dol}}(X/S)$ and

$$(\mathrm{H}^i_{\mathrm{Dol}}(X/S), \vartheta) \cong \mathrm{R}^i f_{\mathrm{Dol}\,*}\mathscr{O}_{X_{\mathrm{Dol}}}$$

This is precisely the Higgs structure induced from the Kodaira-Spencer map of f, $\kappa : \Omega_S^\vee \to \mathrm{R}^1 f_*\Omega^\vee_{X/S}$. We construct this Higgs field as follows: for every (p, q), cup product with the associated section in $\mathrm{R}^1 f_*\Omega^\vee_{X/S} \otimes \Omega_S$ gives an $\mathscr{O}_S$-linear map

$$\mathrm{R}^p f_*\Omega^q_{X/S} \to \mathrm{R}^{p+1} f_*\Omega^{q-1}_{X/S} \otimes_{\mathscr{O}_S} \Omega_S$$

This yields an $\mathscr{O}_S$-linear map $\vartheta : \mathrm{H}^i_{\mathrm{Dol}}(X/S) \to \mathrm{H}^i_{\mathrm{Dol}}(X/S) \otimes_{\mathscr{O}_S} \Omega_S$ with $\vartheta \wedge \vartheta = 0$.

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