SPECTRAL SEQUENCES FOR SHEAF COHOMOLOGY

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Abstract. These notes were prepared for a seminar on *coherent sheaves and their parameter spaces* at the University of Pennsylvania in Fall 2025. We construct the hypercohomology spectral sequence and the Grothendieck spectral sequence, and outline important consequences.

0.1. **Preliminaries.** We begin by recalling the basics of spectral sequences.

Definition 0.1. Given an abelian category $\mathcal{A}$, a cohomological spectral sequence is a sequence $\{E_r^{pq}, d_r^{pq}\}$ of objects E_r^{pq} and differentials $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ satisfying

(a) $d_r \circ d_r = 0$,

(b) $E_{r+1}^{pq} \cong \ker d_r^{pq} / \operatorname{im} d_r^{p-r,q+r-1}$.

A spectral sequence stabilizes if $E_r^{pq} = E_{r+1}^{pq} = \cdots$ for all sufficiently large r; this stable value is denoted E_∞^{pq}. A spectral sequence is *weakly convergent* if E_∞^{pq} is the associated graded of a filtered object; that is, if there exists an object $H^\bullet$ with a decreasing filtration $F^\bullet$ such that

$$E_\infty^{pq} = F^p H^{p+q} / F^{p+1} H^{p+q} = gr^p H^{p+q}$$

This is written $E_r^{pq} \implies H^{p+q}$. A spectral sequence *converges* if the filtration is exhaustive and separated: $\cap_p F^p H^n = 0$ and $\cup_p F^p H^n = H^n$.

Example 0.2. We recall the construction of a spectral sequence on a filtered complex. Let $C^\bullet$ be a complex in $ch(\mathcal{A})$ with a decreasing filtration $F^\bullet$ satisfying $d(F^p C^n) \subset F^p C^{n+1}$. We define r-almost cycles and r-almost boundaries

$$Z_r^{p,q} := \ker\big(d : F^p C^{p+q} / F^{p+1} C^{p+q} \longrightarrow F^p C^{p+q+1} / F^{p+r} C^{p+q+1}\big)$$

$$B_r^{p,q} := \operatorname{im}\big(d : F^{p-r+1} C^{p+q-1} / F^{p-r+2} C^{p+q-1} \longrightarrow F^p C^{p+q} / F^{p+1} C^{p+q}\big)$$

and then we let $E_r^{pq} = Z_r^{pq} / B_r^{pq}$, with differentials d_r^{pq} given as the restriction of cochain differentials d. If this filtration is exhaustive and separated, then

Lemma 0.3. *$\{E_r^{pq}, d_r^{pq}\}$ is a spectral sequence with $E_r^{pq} \implies h^{p+q}(C^\bullet)$ and*

$$E_0^{pq} = gr^p C^{p+q}$$

$$E_1^{p,q} = h^{p+q}(gr^p C^\bullet)$$

0.2. **Filtered Double Complexes.** Let $C^{\bullet,\bullet}$ be a first quadrant double complex in $ch^2(\mathcal{A})$. Recall that the totalization $tot(C)^\bullet$ is the complex defined by

$$tot(C)^n = \bigoplus_{i+j=n} C^{i,j}$$

with differential $d^n|_{C^{ij}} = d_1^{ij} + (-1)^i d_2^{ij}$. To orient ourselves, suppose that d_1 travels in the horizontal direction and d_2 travels in the vertical direction. Note that there are two natural filtrations on $tot(C)^\bullet$:

$$F_{(1)}^p tot(C)^n = \bigoplus_{r \geq p} C^{r,n-r}$$

$$F_{(2)}^p tot(C)^n = \bigoplus_{r \geq p} C^{n-r,r}$$

The former is a filtration in the vertical direction, and the latter in the horizontal direction. We define complexes $h_1^{\bullet,j}(C)$ and $h_2^{i,\bullet}(C)$ by

$$h_1^{\bullet,j}(C) = \ker d_2^{ij} \,/\, \operatorname{im} d_2^{i,j-1}$$

$$h_2^{i,\bullet}(C) = \ker d_1^{ij} \,/\, \operatorname{im} d_1^{i-1,j}$$

Caution 0.4. The former is the complex arising from taking the j-th vertical cohomology of the columns and hence travels in the horizontal direction. The latter is the i-th horizontal cohomology of the rows and hence travels in the vertical direction.

Proposition 0.5. *By 0.3, there are spectral sequences* ${}^{(1)}E_r^{pq}$ *and* ${}^{(2)}E_r^{pq}$ *with*

$$\begin{aligned} {}^{(1)}E_0^{pq} &= C^{pq} & {}^{(2)}E_0^{pq} &= C^{qp} \\ {}^{(1)}E_1^{pq} &= h_1^{pq}(C) & {}^{(2)}E_1^{pq} &= h_2^{qp}(C) \\ {}^{(1)}E_2^{pq} &= h^p(h_1^{\bullet,q}(C)) & {}^{(2)}E_2^{pq} &= h^p(h_2^{q,\bullet}(C)) \end{aligned}$$

both of which converge to $h^{p+q}(tot(C)^\bullet)$.

Remark 0.6. In what follows, we primarily use ${}^{(2)}E_2^{pq}$.

0.3. **Hypercohomology.** Recall that if $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor between abelian categories, where $\mathcal{A}$ has enough injectives, and $C^\bullet$ is a complex in $ch(\mathcal{A})$, the n-th right hyperderived functors of F are defined by

$$\mathbf{R}^n F(C^\bullet) = h^n(F(tot(I)^\bullet))$$

where $C^\bullet \to I^{\bullet,\bullet}$ is an injective resolution.

Proposition 0.5 extends to a spectral sequence computing hyperderived functors.

Proposition 0.7. *There is a spectral sequence* E_r^{pq} *such that*

$$E_2^{pq} = R^p F(h^q(C^\bullet)) \implies \mathbf{R}^{p+q} F(C^\bullet)$$

which allows us to compute hyperderived functors in terms of derived functors.

Sketch of Proof. Take I to be a Cartan-Eilenberg resolution of C.[1] The spectral sequence ${}^{(2)}E_2^{pq}$ in 0.5 for the double complex $F(I)^{\bullet,\bullet}$ yields the desired spectral sequence. The main idea is that $h^q(C^\bullet) \to h_2^{q,\bullet}(I)$ is an injective resolution, so

$$R^pF(h^q(C^\bullet)) = h^p(F(h_2^{q,\bullet}(I)))$$

by definition. Since I is Cartan-Eilenberg, there is a natural isomorphism

$$h^p(F(h_2^{q,\bullet}(I))) = h^p(h_2^{q,\bullet}(FI))$$

and we observe that the right-hand side is precisely ${}^{(2)}E_2^{pq}$. Then 0.5 implies

$${}^{(2)}E_2^{pq} \implies h^{p+q}(tot(FI)^\bullet)$$

noting that by the additivity of F the right-hand side is $\mathbf{R}^{p+q}F(C)$. □

0.4. **Grothendieck Spectral Sequence.** An important consequence of this result is a spectral sequence for the composition of derived functors. Consider left exact additive functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ where $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, and such that F sends injective objects to G-acyclic objects ($\star$).

Remark 0.8. In the following derivation, we employ the machinery of total derived functors and derived categories. Recall that the total derived functor

$$RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$$

is a functor between the (bounded below) derived category of $\mathcal{A}$ and that of $\mathcal{B}$. There is a natural isomorphism of functors $h^i(RF) \cong R^iF$ and $R(G \circ F) \cong RG \circ RF$.

Let K be an object of $\mathcal{A}$, and consider the total derived functor $RF(K)$, which has a representative in $ch(\mathcal{B})$. Applying the hypercohomology spectral sequence,

$$\begin{array}{ccc} E_2^{pq} = R^pG(h^q(RF(K))) & \Longrightarrow & h^{p+q}(RG(RF(K))) \\ \| & & \| \\ R^pG(R^qF(K)) & & R^{p+q}(G \circ F)(K) \end{array}$$

We define the above to be the *Grothendieck spectral sequence* for F and G.

0.5. **Leray Spectral Sequence.** Let $f : X \to Y$ be a continuous map of topological spaces and consider the induced functors

$$\mathrm{Shv}(X) \xrightarrow{f_*} \mathrm{Shv}(Y) \xrightarrow{\Gamma} \mathrm{Ab}$$

Note that $\Gamma(X, \mathscr{F}) = \Gamma(Y, f_*\mathscr{F})$. As shown previously, the category of sheaves of abelian groups on X contains enough injective objects. The following lemma shows that f_* preserves injective objects in view of the adjunction

$$f^{-1} \dashv f_*$$

[1] A Cartan-Eilenberg resolution for an object exists provided that $\mathcal{A}$ has enough injectives.

with $f^{-1} : \mathrm{Shv}(Y) \to \mathrm{Shv}(X)$ exact. This last fact follows since a sequence of sheaves is exact if and only if it is exact on stalks and $f^{-1}(\mathscr{G})_x \cong \mathscr{G}_{f(x)}$.

Lemma 0.9. *If $L : \mathcal{B} \to \mathcal{A}$ is exact with right adjoint R, then R preserves injectives.*

Proof. Suppose that A is an injective object of $\mathcal{A}$. Given a monomorphism $B \to B'$ and a morphism $B \to R(A)$, apply the exact functor L to obtain a monomorphism $L(B) \to L(B')$ and a morphism $L(B) \to LR(A)$. The latter maps to A via the counit of the adjunction $LR \to \mathrm{id}_{\mathcal{A}}$. Since A is injective, there exists an extension $L(B') \to A$. Applying R yields $RL(B') \to R(A)$. This gives a compatible map $B' \to R(A)$ via the unit, which completes the proof. □

Thus the image of f_* is Γ-acyclic. Consequently, hypothesis $(\star)$ is satisfied. Therefore the Grothendieck spectral sequence for the composition yields

$$E_2^{pq} = H^p(Y, R^q f_* \mathscr{F}) \implies H^{p+q}(X, \mathscr{F})$$

which is the *Leray spectral sequence.*

Example 0.10. (Higher Direct Images) Consider the composition

$$\mathrm{Shv}(X) \xrightarrow{i} \mathrm{PShv}(X) \xrightarrow{f_*} \mathrm{PShv}(Y) \xrightarrow{sh} \mathrm{Shv}(Y)$$

where $i : \mathrm{Shv}(X) \to \mathrm{PShv}(X)$ is the forgetful functor and $sh : \mathrm{PShv}(Y) \to \mathrm{Shv}(Y)$ denotes sheafification. Since the forgetful functor is right adjoint to sheafification, it is left exact. Moreover, at the level of presheaves, the functors f_* and sh are *exact*. Thus for a sheaf $\mathscr{F}$ on X,

$$R^q f_* \mathscr{F} = R^q(sh \circ f_* \circ i)(\mathscr{F}) = (sh \circ f_*)(R^q i \mathscr{F}) \qquad (1)$$

For an open $U \subset X$, consider the composition $\mathrm{Shv}(X) \to \mathrm{PShv}(X) \to \mathrm{Ab}$ of $\Gamma_U \circ i$. Then

$$H^q(U, \mathscr{F}) = R^q(\Gamma_U \circ i)(\mathscr{F}) = \Gamma(U, R^q i \mathscr{F}) \qquad (2)$$

again by the exactness of Γ_U for presheaves. We conclude that $R^q f_* \mathscr{F}$ is the sheafification of the presheaf

$$U \mapsto H^q(f^{-1}U, \mathscr{F})$$

by (1) and (2).

0.6. **Čech-to-cohomology spectral sequence.** Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be a covering of a topological space X. We define a functor $\check{H}^0(\mathfrak{U}, -) : \mathrm{PShv}(X) \to \mathrm{Ab}$ by

$$\mathscr{F} \mapsto \mathrm{eq}\left(\prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j) \right)$$

One can verify that $\check{H}^0(\mathfrak{U}, -)$ is additive and left exact. We define

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) \coloneqq R^p\check{H}^0(\mathfrak{U}, \mathscr{F})$$

to be the p-th Čech cohomology of $\mathfrak{U}$ with values in the presheaf $\mathscr{F}$. Consider the composition of functors

$$\mathrm{Shv}(X) \xrightarrow{i} \mathrm{PShv}(X) \xrightarrow{\check{H}^0(\mathfrak{U},-)} \mathrm{Ab}$$

Note that for a sheaf $\mathscr{F}$, $\check{H}^0(\mathfrak{U}, \mathscr{F}) = \mathscr{F}(X)$. The inclusion i has an exact left adjoint given by sheafification, 0.9 tells us that i sends injective objects to injective objects. Consequently, hypothesis ($\star$) is satisfied, and there is a Grothendieck spectral sequence

$$\check{H}^p(\mathfrak{U}, R^q i\mathscr{F}) \implies H^{p+q}(X, \mathscr{F})$$

abutting to the standard cohomology of X valued in $\mathscr{F}$.

Exercise 0.11. If $\mathfrak{U} = \{U, V\}$, show that this spectral sequence degenerates at E_2 and yields the Mayer-Vietoris sequence:

$$0 \to H^0(X, \mathscr{F}) \to H^0(U, \mathscr{F}) \oplus H^0(V, \mathscr{F}) \to H^0(U \cap V, \mathscr{F}) \to H^1(X, \mathscr{F}) \to \cdots$$

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